

# Mathematics Classrooms

## Students' Activities and Teachers' Practices

Fabrice Vandebrouck (Ed.)



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## MATHEMATICS CLASSROOMS: STUDENTS' ACTIVITIES AND TEACHERS' PRACTICES



# Mathematics Classrooms: Students' Activities and Teachers' Practices

*Edited by*

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BARBARA JAWORSKI

## PREFACE

This book presents unique insights into a significant area of French research relating the learning and teaching of mathematics in school classrooms and their development. Having previously had only glimpses of this work, I have found the book fascinating in its breadth of theory, its links between epistemological, didactic and cognitive perspectives and its comprehensive treatment of student learning of mathematics, classroom activity, the work of teachers and prospective teacher development. Taking theoretical perspectives as their starting points, the authors of this volume present a rich array of theoretically embedded studies of mathematics teaching and learning in school classrooms.

The book charts the use of a theoretical/methodological perspective called *The Double Approach*, a didactic and ergonomic approach for the analyses of teaching practices (Robert & Rogalski, 2002). This approach is concerned simultaneously with the design of teaching and with its practical, ergonomic (work-based) contribution to students' learning of mathematics in classrooms. It seeks to address associated issues widely and in their full complexity recognising institutional dynamics and constraints, the impact of social and cultural perspectives and interweaving layers of activity.

The term "activity" is ubiquitous throughout, taking on two kinds of meaning, in one sense referring to the actions of students and teachers in the classroom and, in parallel, referring also to *activity* as in *Activity Theory*, a complex dynamic encompassing the wholeness of classroom learning and teaching; as Leont'ev has expressed it, "the non-additive, molar unit of life ... a system with its own structure, its own internal transformations, and its own development" (Leont'ev, 1979, p. 46). The two senses are deeply entwined in the ways activity is addressed. Thus, it is not surprising that one of the foundations of the Double Approach, its "organising framework" is the sociohistorical theory of Vygotsky, and followers such as Leont'ev. In Chapter 1, Janine Rogalski writes, "the object of study consists of the activity of an individual subject with individual motivations, within a specific situation. When the subject is a teacher, it is not the "properties" or "functioning" of the teacher's position that is at issue. ... Rather the issue involves questions of diversity among teachers, and the development and emergence of their individual professional competencies" (p. 3). The focus on the individual subject ("as a person-subject rather than as a didactic subject," *ibid.*) is perhaps somewhat more surprising, especially since it leads the authors to consider a Piagetian approach of epistemological genetics alongside Vygotsky's sociohistorical framework. The surprise is in the juxtapositioning of theories of Piaget and Vygotsky of which others scholars have been cautious, if not dismissive, due to the (supposed) incommensurability of these theoretical perspectives (see e.g., Lerman,



1996). More recently Lerman (2013) wrote “In general, drawing on more than one theoretical perspective needs some work in order to ensure that the perspectives are coherent together.” Regarding the complementarities of the two areas of theory, Rogalski writes: “In particular, the Piagetian theory looks “from the student’s side” at epistemological analyses of the mathematical objects in play, while the Vygotskian theory takes into account the didactic intervention of the teacher, mediating between knowledge and student in support of the students’ activity” (p. 23). Just one of the exciting aspects of these authors’ use of the Double Approach is to see how this theoretical juxtapositioning leads to analyses of teacher and classroom activity which make sense for the researchers and those who learn from this research.

So, what exactly is meant by the *Double Approach*? Aline Robert and Christophe Hache (Chapter 2) weave the abstract theory (above) with theoretical frameworks relating to teaching and learning in classrooms and general methodologies that follow from these frameworks. They write, “... we seek to measure the gap between the activities of students applying their knowledge (during its acquisition) analyzed *a priori*, and the activities that may actually have taken place during a regular lesson” (p. 62). At a simple level, we see an analytic progression from epistemological analysis in a mathematical topic, through a didactic analysis relating to the design of teaching, into analyses of classroom activity and inter-relations between teachers and students with, last but not least, analyses of student activity and understanding. The progression is not linear (as my list might suggest); the research lens may focus in any of these areas, or zoom out to address complex inter-relationships between them. Thus the programme is ambitious. The reader is taken through subdivisions of “the world of the study:” we read of student activity, the transition from designed task to student activity with task, levels of conceptualization (related to Vergnaud’s, 1990, ‘conceptual fields’), the nature of concepts and students’ progression with concepts in terms of generalization and formalization, the knowledge of teachers and design of tasks, wider issues in terms of systemic demands or emotional, personal and social factors. For example, relationships between didacticians and teachers are addressed, ways in which teachers adopt or adapt didactic designs, the ‘work’ of the teacher in the classroom, teacher speech patterns and representations of mathematical concepts and their relationships to student activity. Consideration of the profession of teaching and roles in teachers’ work lead to questions of teaching development and the education of new teachers.

In presenting theory and methodology in these areas, the authors move to and fro between the cognitive and the sociocultural frames so the reader is faced with challenges in making sense of the complexities involved. Unsurprisingly, this ambitious enterprise raises many questions for the reader, not least as to how theoretical complexities are translated into practice in schools and classrooms, how teachers work with researchers (or independently with these perspectives), and how researchers address inter-relationships between observations of student conceptualisation, teachers’ didactic processing in design of activity and the wider frames of educational and sociocultural impact. For example, as the authors

acknowledge, “if it is difficult to analyse teaching in relation to learning, it is even more difficult to have legitimate evidence of it” (p. 58).

These questions are addressed variously in the (nine) chapters which follow in which we see the elements of the Double Approach in action with differing zooms of the research lens. Authors present a variety of methodological approaches with analyses of classroom settings, design of tasks, mathematical topics, teachers’ intentions, student responses, imposed constraints and the degree of ‘leeway’ experienced by the teachers. For example, in Chapter 3, Eric Roditi discusses the tasks offered by four teachers in similar sixth grade classes on the topic of multiplication of decimal numbers. We read of the nature and choice of tasks, their levels of cognitive demand and their relation to curriculum guidance on the topic, suggested class time, and expectations of professional practice discerned through classroom observations and interviews with teachers. *A priori* expectations of students’ activity according to designed tasks is compared with student outcomes in the tasks. Despite the commonalities of designed tasks in the four classrooms, research emphasised and categorized the variability of classroom activity depending on the ways in which individual teachers worked with students in their class.

The work of teachers and its relation with the realities of situation and context is central to methodologies employed. The relationship between teaching practices and student learning is a recurring theme. Julie Horoks writes in Chapter 6 (p. 135), “Naturally we are not questioning the teacher’s work, and we will consider the different components of his/her job to explain certain choices made for his/her class.” Horoks describes the use of classroom video recordings to reveal teachers’ use of tasks focusing on similar triangles and to relate students’ degrees of success with these tasks to the ways in which the tasks were used in the classroom. Monique Chappet-Paries, Aline Robert and Janine Rogalski, in Chapter 4, focus on classroom activity around the theorem of Pythagoras, analysing a teacher’s speech patterns to gain insight into invariants in the teacher’s practice and ways in which these invariants impact on students. The idea of a “teaching scenario” – a sequence of lessons and exercises around a mathematical topic such as decimal numbers, similar triangles or Pythagoras’ theorem (studied *a priori*) – is a common theoretical construct. For example, in Chapter 7, Aurélie Chesnais discusses the implementation of the same teaching scenario (about orthogonal symmetry) by two experienced teachers in order to study regularities and variability of practices between teachers, as well as the relationship between teaching practices and student learning.

The major theme of relations between *a priori* analyses of tasks, teachers’ implementation of tasks in the classroom and students’ take-up of tasks is considered in later chapters with an added dimension, that of the use of electronic resources. In Chapters 8 and 9 these are *Electronic Exercise Bases*, consisting of mathematics exercises within an environment, which includes different types of suggestions, aids, tools (graphs, calculators, etc.), lesson reminders, as well as explanations, answer analyses or complete solutions. The scenarios here are designed around the electronic environment and its use by students with a study of,

for example, how the designed situations influence the students' activity. Research reveals that the expected activity is not always the activity developed by the students and emphasizes the difficulty for students to regulate their activity while interacting with the software without teacher intervention. As well, studies address the impact of the electronic resources on the day-to-day activities of teachers and on teachers' evolving classroom practices. Chapter 10 compares activity in a dynamic geometry environment with that in a pencil and paper environment to analyze how the tasks designed for ICT environments differ (or not) from those of non ICT ones. Research explored the differences in classroom management, including ways in which the teacher assisted students, in order to understand their possible impact on students' activities.

I have given these very brief sketches of the focus of various chapters to illustrate or exemplify the pervasive themes of the book in addressing classroom complexity and the deeply inter-related nature of teaching-learning activity. Each study presents different facets of design, implementation and impact of scenarios within the real constraints of classrooms and the personal and social influences which surround classroom interactions. In the final chapter, Maha Abboud-Blanchard and Aline Robert reflect on the earlier chapters to distil elements of their findings which offer insights that are useful in considering the education of mathematics teachers, and of those who will train mathematics teachers. They ask the question, "who should be trained first – the teacher or the teacher's trainer?" (p. 235). This leads to their setting out a training programme for the trainers of mathematics teachers. They acknowledge that this is speculative and that associated research is yet to be undertaken. It nevertheless points to the ambitious scope of the book and the broad programme of research it charts.

Throughout this book the reader is made aware of many unanswered questions and challenged to consider associated theoretical and methodological issues. There is nevertheless an internal consistency and coherence to this work which revolves around the Double Approach. For English-speaking communities who have lacked opportunity to access the French literature the book opens up a wealth of new ways of thinking about and addressing unresolved issues in mathematics learning, teaching and teacher education. I recommend it wholeheartedly!

References are included in the general bibliography of the book.

FABRICE VANDEBROUCK

## INTRODUCTION

This book presents several works in the field of mathematics didactics revolving around secondary school and university teaching. The specificity of the research studies in question is that they attribute as much importance to the actors (students and teachers) as to the mathematics and the school situation. These studies fit well in the very general framework of Activity Theory.

The presented researches aspire to analyze what is at play in a mathematics classroom, by varying the school situations, the environments, the contents, the teachers and the classrooms. The main objective is to study, understand, and even interpret the links between the teaching of a given mathematical content and the corresponding student learning. We seek to highlight regularities and variations of these processes in order to better understand students' acquisitions, and interpret the teachers' practices. The work as a whole leads to inferences which can contribute to the professional development of teachers by widening the range of possible activities for each teacher.

The general framework of Activity Theory, with associated development theory, is described in chapter 1, and we directly clarify how this work fits in this framework. The analyses of students' in-class activities, as they are organized by the teachers, provide us with data which allow us to tackle teachers' practices and approach students' learning: the general theory accounts for this focus and the corresponding reality splitting. Nevertheless, the way activities are assigned to mathematics and school situations is not very present in the framework of Activity and development theories. Therefore, the necessary theoretical and methodological complements are presented in chapter 2.

The main concern of this book is however not theoretical, even though its specificity borrows elements from Activity Theory and development theories which complement typically didactical tools. We seek to assign to the singular subjects (students and teachers) their place within the didactical relationship, even though the affective and social factors are not directly accounted for, despite their high importance. We develop the means to collect and analyze in a significant way, adapted to our project, data about teaching and learning allowing us to interpret the relationship between the two.

All the research studies of this book follow a common methodology presented in chapter 2, but involve, of course, indispensable adaptations which are introduced gradually. They pertain to the teaching of mathematics in middle school, high school, or the first two years of university. Some works are the fruit of individual research<sup>1</sup> and handle a small number of cases in an exhaustive manner, often over quite short periods of time. Others works are clusters of research studies or the fruit

of collective research based on larger data and oriented more directly towards results which are relevant for the general research question of the book. In any event, there is always a limit to the hasty generalizations of results. Hence, there are neither definitive results nor (even less) prescriptions in our discourse.

In chapters 1 and 2, we present the theoretical frameworks and the tools used in the book, while stating the specificity of our research. Chapters 3 and 4 are concerned with the results about teachers' practices in "ordinary" classrooms. They highlight the stability of teachers' practices and also account for the diversity and variability between the teachers. Chapter 5 deals with teaching manuals and shows that exercises proposed in these manuals do not offer the teachers opportunities to diversify their student activities. Chapters 6 and 7 refer more directly to teachers' practices in relation with students' activities. Chapter 8 focuses on the activity of students in a specific teaching situation in a computerized environment. Chapters 9 and 10 deal with teachers' practices in computerized environments, in particular the comparison of teachers' activities in different environments. Chapter 11 is a large scale study about teachers' practices and the factors related to the regularity and variability of the practices. Last, chapter 12 comes as a synthesis of the book with an opening on professional development of teachers.

The different chapters can be read in a relatively independent way. In particular, it is not necessary to complete an exhaustive reading of chapters 1 and 2 in order to read the other chapters ... and vice versa!

#### NOTES

- <sup>1</sup> All the researchers who contributed to this book, apart from Aurélie Chesnais, Eric Roditi and Janine Rogalski, are members of the Laboratoire de Didactique André Revuz (LDAR) at Paris Diderot University

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# **1. THEORY OF ACTIVITY AND DEVELOPMENTAL FRAMEWORKS FOR AN ANALYSIS OF TEACHERS' PRACTICES AND STUDENTS' LEARNING**

## **INTRODUCTION**

The goal of this chapter is to propose a theoretic framework to analyze the structured activities of teachers and their students, and to provide support for some inferences regarding teachers' training in professional competencies' and students' acquisition of knowledge in specific disciplines.

The organizing framework is that of the theory of activity, which was established by Leontiev, enriched through a line of research originated by Vygotsky, and then exploited and developed within the field of ergonomic psychology (Leplat, 1997; Rogalski, 2004). Its fundamental components are:

- the distinction between task and activity;
- the double point of view, taking into account both the situation and the subject of the action; and
- the system of double regulation of activity, in which determining factors, and the effects of the activity, influence situational components as well as the subject. This regulation is not only retroactive, but also proactive, as a goal-oriented activity is affected when subjects adapt their actions in an attempt to produce the desired results.

Within this theoretic framework, the object of study consists of the activity of an individual subject, with individual motivations, within a specific situation. When the subject is a teacher, it is not the “properties” or “functioning” of the teacher’s position that is at issue here (as would be the case for a *stricto sensu* didactic perspective, which we could define as the “science of didactic processes”). Rather, the issue involves questions of diversity among teachers, and the development and emergence of their individual professional competencies. Equally relevant are considerations of the student as a person-subject, rather than a didactic subject. All this leads us to consider the Piagetian approach of epistemological genetics, together with Vygotsky’s socio-historical framework, as they relate to individual development.

Taking into account the effects of the activity on the subject is an aspect of the developmental and constructivist dimension of the theory of activity (TA). Our focus is on the activity, on its determining factors and on its effects as they relate to teaching mathematics. We are particularly interested in the activity’s effects on a teacher’s development of professional competencies, and on a student’s mathematical conceptualization. Interpreting TA within the theoretical frameworks

of Piaget and Vygotsky enriches our approach, by defining the developmental dimension of the double regulation in terms of factors and effects, temporality, and the role of psychological tools (Vygotsky) and cognitive tools (Piaget). We include within the Vygotskian framework Bruner's findings on mediation (Wood, Bruner, & Ross, 1976), which add to our understanding of the didactic intervention of a teacher in class. We conclude with a discussion of these theoretical frameworks in order to define the tools we will use in our analysis of teaching practices and student activities in mathematics.

### THE THEORY OF THE GOAL-ORIENTED ACTIVITY

The theory of activity was developed by researchers who followed Vygotsky in studying the psychology of work (later called "ergonomic psychology"). The theory was then used in professional didactics, before being "articulated" with a didactical approach to mathematics teaching, in the so-called "double approach" (Robert & Rogalski, 2002, 2005; Robert, chapter 2). The theory involves goal-oriented and motivated activities. By their actions, subjects aim to achieve task goals, and their actions are driven by motivations of the activity.

We will describe the following elements, all of which are essential to our objectives: the task-activity distinction, task structure, the various ways to analyze an activity, and the connection between the subject and the situation within the model of double regulation of activity. We will also indicate how this theoretical framework allows us to analyze the structure of teacher and student activities.

#### *Task and activity*

The task-activity distinction is central to the theory of activity. The activity relates to the subject,<sup>2</sup> while the task relates to the objects of the action. The definition of a *task*, as proposed by Leontiev (1975, 1984) and developed by Leplat (Leplat and Hoc, 1983; Leplat, 1997) is the "goal to be attained under certain circumstances." The *activity* is what a subject engages in during the completion of the task. This includes not only external actions, but also inferences, hypotheses, decisions, and actions the subject decide *not* to take. The activity also includes the subject's time management and personal state – workload, fatigue, stress, enjoyment of work – as well as interactions with others within the work situation. We will first consider the task, and describe its essential characteristics. We will then examine the activity developed in response to the task.

#### *Structure of a task*

The *task object* is that which is to be transformed or studied. Tasks involving material objects were originally the most studied by ergonomic psychology. Tasks for which the "objects" include human individuals (service professions, therapeutic work, teaching) or for which the goal is to learn and acquire tools for thought (being a student) require a more complex analysis. For the teacher, the goal to be

attained is often described in procedural terms, with action verbs: “Teach the concept of length measurement to elementary school students,” “correct a math test,” “follow the curriculum.” Goals can also be stated with reference to the student-knowledge relationship: “Have the student acquire the concept of length and linear units,” “Have the student represent functions as mathematical objects and tools.” For the student, the task is defined by the teacher’s statement, and the requirements of mathematical work.

### *Tasks and sub-tasks*

In a complex situation, the goal to attain consists of various sub-goals, whose achievement order is more or less constrained. For example, “introduce students to the concept of functions,” in ninth grade, involves making documentation choices, creating lesson plans that cover one or more class sessions, defining the student tasks, conducting the in-class activities, and finally evaluating students’ acquired knowledge.

### *The structure of the task involves transitions between the intentions of the prescribed task and the actual task as implemented*

In a workplace (in the teacher’s case) or learning environment (in the student’s case), the subject responds to tasks assigned by a prescriber, with the framework for completion defined by the desired results and the permitted resources. This constitutes a prescribed task. But an *activity* is not a direct response to a prescribed task. The task is first redefined by the subject. To complete this task, the subject must form a representation of the task, allowing or forbidding possibilities (not always consciously), lifting or imposing restrictions, and using evaluation criteria that may differ from those of the prescription. This constitutes the *effective task*, to which the subject’s activity represents a response. Misunderstandings in teaching are an expression of differences between the task anticipated by the teacher, and the task responded to by the student.

The gap between prescribed and effective tasks is inherent to the existence of two viewpoints: That of the task prescriber, and that of the task completer. The task the subject completes can differ from the assigned task for various reasons: Because the subject lacks motivation to engage in the desired actions, because the subject lacks the necessary competencies, because the subject constructed an inappropriate representation of the task, or even because of a divergence between the intended and prescribed tasks. The effective task is revealed by the subject’s activity.

### *Analysis of the activity*

In work or training situations, activity is oriented towards the completion of the task. Observable actions that permit an analysis are, first, operations on the objects of action, regardless of the aim of the research. This explains why the analysis of the activity relies on a preliminary analysis of the task, which can be understood as a psychological task analysis (Vicente, 1999) that relies on domain expertise.



However, the activity includes more than simply actions on “what to do,” and includes other personal factors. For example, a teacher can assign different sets of problems from one year to another, not only because of the effect on students, but also to maintain personal motivation and avoid repetition or fatigue.

The analysis of a student task requires a didactician’s mathematic expertise in order to identify what a student can do to effectively complete it. This is the aim of the *a priori* analysis presented in Chapter 2. The analysis of the teacher’s task is more delicate. It is a largely discretionary task for which there is no defined procedure to follow (Leplat, 1997, p. 21). How to identify a strategy that would lead to the desired goal remains an open question, as there is no commonly accepted definition of an “expert teacher.”<sup>3</sup> For this analysis of teachers’ tasks and activities, we refer to a model of teaching as management of a dynamic human environment (Rogalski, 2003), in which the teacher mediates (Wood, Bruner, & Ross, 1976) between the student and the knowledge to be acquired (Robert & Rogalski, 2005), and in which language plays a central role (Pariès, Robert, & Rogalski, 2005).

### *The subject and the situation*

The theory of activity depends on two other key concepts: The subject and the situation. We are interested in an *individual subject*, who has intentions and competencies (potential resources and personal constraints). Within this framework, subjects do not identify with their role, even though they may be constrained by legal and other responsibilities that act on the teacher. We can look both for commonalities between subjects and for specific aspects of their activities: What factors and organizational aspects do they share? What are the individual differences between them?

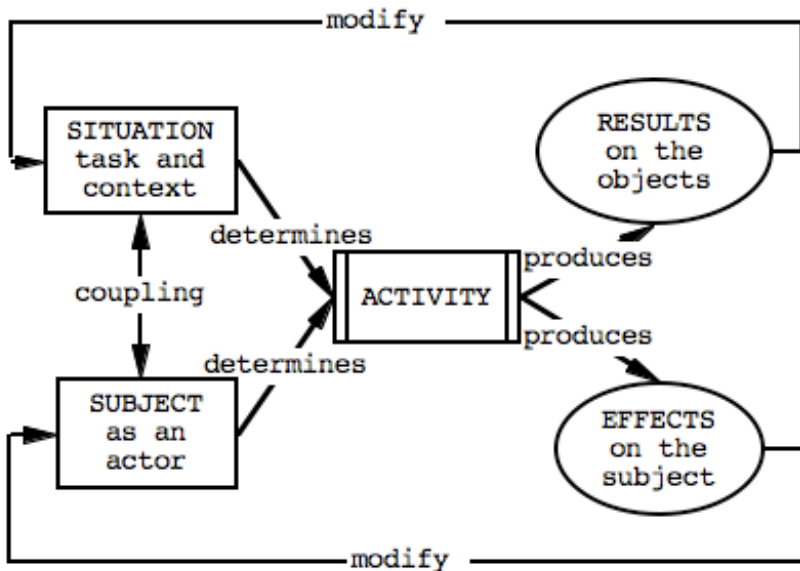
Whether students or teachers, subjects are not the sole masters of their goals or methods. They act within a *work or training situation*, which consists of a system of resources and constraints. Within this system, the teacher completes a set of tasks, which we can more globally consider to be a mission (the discretionary dimension of the task), tied to a prescriber (employer, supervisor) by a partially implicit contract. The teacher is acting within a context where students encounter multiple interventions (parents, teachers of other subjects, etc.) and within a process that continues during students’ entire schooling. The student’s situation is not limited to the tasks prescribed by the teacher under a didactic contract, but includes the social and familial environment.

We will now present the model of double regulation of activity, which can be related to issues of learning and development, as defined through the theories of Piaget and Vygotsky, and expanded by Vergnaud. Later on, we will defend the complementarity of Piaget and Vygotsky.

*The model of double regulation of activity*

The concept of regulation reflects the fact that the activity modifies the state of the situation as much as the state of the actor. The situation is a determining factor of the activity, and is simultaneously itself modified by the activity. This modification primarily affects the object of the activity, but can also include modification of resources and constraints. Subjects, too, both determine the activity and are modified in turn by their own activity. The situation can affect their potential for knowledge and action (competencies), their physical state (tired, sleepy, etc.), or their emotional state (happy, bored, anxious, etc.).

Figure 1 presents a schematic diagram of how this system of double regulation relates to the system of situational and subject determinants.



*Figure 1. The double regulation includes a co-determination of the activity by situational and subject properties, as well as a double modification of the situation and the subject that is created by the results and effects produced (and by their agreement with expectations and acceptable outcomes).*

This regulation can be considered in terms of short-term adjustments to action and “local” learning (such as learning how to find the inverse image of a function on a graph), or in terms of long-term development of a subject (understanding the concept of a function). The model of double regulation fits directly with the constructivist theories of Piaget and Vygotsky. It also sheds light on the issue of didactic intervention, by considering situational properties as potential producers of

learning and development. Before considering this further, we must first pause to define the specific activities of the teacher and student, respectively.

*A framework for defining student and teacher activities*

The student, whether as an individual, set of students, or class as a whole, is a central determining factor of the teacher's activity. The choice of lesson plans (student task organization), and the unfolding of these plans in class, depends on prior knowledge of students, as well as on the possible actions the teacher believes to be possible in class. The teacher's didactic interventions in class depend on students' individual or collective activity. Completing a task produces a return effect on the teacher's activity, with an eventual adjustment both of the proposed tasks and of the teacher's own activity. Students' behavior can also contribute to the effects on the teacher, inducing fatigue, enjoyment, etc.

The teacher determines the activity of the student through the assigned mathematical tasks. During the completion of a task, the teacher mediates between the students and the mathematical concept to be acquired. This mediation can consist of assistance in getting started, procedural or constructivist assistance in completing the task, evaluation of the final product, identification of the concept in play, etc. The teacher can also participate in the construction of a student's reflexivity (for example, by demonstrating how to solve problems) and intervene in the constructivist dimension of the student's activity. Chapter 2 will explore this question.

COMPARISON OF THE THEORIES OF PIAGET AND  
VYGOTSKY ON DEVELOPMENT AND LEARNING

Piaget and Vygotsky each elaborated theoretical frameworks for understanding children's (and, more generally, humans') developmental processes. We will first present each researcher's scientific objectives, then the relevant elements of Piagetian constructivism, and finally Vygotsky's theoretical contributions. Putting these two frameworks in perspective highlights their commonalities, which include factors of development, a long-term perspective, and the role of tools in development (called "cognitive tools" by Piaget and "psychological tools" by Vygotsky).

*Piaget's and Vygotsky's scientific objectives*

For Piaget, the crucial point, distinguishing Piaget from all others in the field and rendering him irreplaceable within the scientific panorama of the 20th century, is his objective of genetic epistemology. The central question of this is how humans acquire knowledge, and how they thereby progress from children to adults capable of contributing to the development of scientific knowledge.

Piaget's aim is to "try to untangle the roots of the diverse varieties of knowledge, beginning with their most elementary forms and following their

development to subsequent levels, including scientific thought” (Piaget, 2005, p.6). He notes that development of knowledge during the evolution of a species could in theory be considered part of this objective, but he chooses to begin with the development of a human child. He insists on the fact that his work has “a psychological dimension, but as a by-product ... the goal is essentially epistemological” (op. cit., p. 7).

Piaget’s viewpoint is therefore that of knowledge development for an *epistemological subject*, which is as much a theoretical construct as the *didactic subject* of mathematical didactics (when it defines teacher and student in terms of their role in the school system). A biologist by training, Piaget always insists on the biological roots of knowledge (Piaget, 1971, 2005, pp. 59-75).

Piaget’s interest in the evolution of the structures of knowledge leads him to neglect a certain number of topics. For example, the topic of the developmental factors of a child (considered as a *psychological subject*) will not be central to Piaget’s work. This is not because Piaget denies the effects of factors that are not internal to the “epistemological subject,” but because his objective is to understand the internal process of development.

Piagetian constructivism claims that knowledge of objects is constructed through actions on these objects, and Piaget’s goal is to demonstrate his approach’s validity on the set of large domains of knowledge. These actions are not limited to physical acts on material objects, as knowledge construction can also occur through mental operations. Observation, for example, is a valid action that affects a subject’s representations.

As for Vygotsky, his goal of theorization is clearly psychological, aiming to theorize the “higher functions” of thought. For him, the subject is a psychological subject, considered from the beginning to be in a social interaction with other subjects who have previously and personally developed “psychological tools”: this enables the development of knowledge. Under this model, knowledge of the world is socially preexisting in children: Their cognitive activities exist within social interactions before they are internalized into a subjective plane. This is the central, and very strong, idea of socio-constructivism: The passage from the inter-individual to intra-individual relies on the construction of psychological tools.

Vygotsky’s focus is therefore profoundly different from Piaget’s, with completely different objectives. Vygotsky’s subject is an individual and social subject, who will construct tools for thought within social interactions. Piaget’s subject is an epistemological subject, for whom the organization of knowledge (rather than mediation or tools) is the issue.

From this starting point, Vygotsky describes in a theoretical fashion the processes of learning and development, without dissociating the two. He will particularly differentiate, within a subject’s “learning-development,” the “everyday” concepts from the “scientific” concepts. Everyday concepts come from the everyday world, where social interactions do not have as a goal the production of an organized conceptual piece of knowledge in children. The acquisition of scientific concepts is accomplished through deliberate didactic interventions (Vygotsky, 1986, chap. 6).

Under Vygotsky's theory, scientific concepts are taught in scholastic institutions, and develop differently from everyday concepts. This deeper theoretical understanding of the evolution of concepts is directly pertinent for all didactics of a knowledge domain.

We highlight these differences between Piaget's and Vygotsky's objectives and central objects as a preface to presenting evidence that their case is not one of two psychologists with conflicting viewpoints, but rather of each making his own specific scientific contribution. Each proposes an original perspective on knowledge construction, as defended by Shayer (2003), for example. We will therefore first go deeper into the framework of Piagetian constructivism, and then describe the theoretical contributions of Vygotskian conceptualism, which are crucial for didactics of science.

### *Piagetian constructivism*

The dominant image of Piagetian constructivism is probably that of a construction of knowledge that is internal to the subject. From this, one could see Vygotskian socio-constructivism as in opposition, taking into account the social dimension that Piaget would supposedly discard. To show that this is simply a question of perspective, we can refer to Piaget himself: "The social group plays ... from a cognitive point of view the same role that the 'population' plays from a genetic point of view. ... In this sense the society is the supreme unit and the individual only achieves his intellectual constructions insofar as he is the seat of collective interactions for which the level [depends] on the society as a whole" (Piaget, 1992, p. 345). The necessity of the social aspect in cognitive development is here clearly affirmed. Piaget successfully integrates the existence of two shifts during development: One associated with the individual as epistemological subject, and the other purely social. But it is the process of organization of knowledge (its structure) that will be central in the research he conducts. This "internal mechanism" of development is conceived in terms of a double regulation, retroactive and proactive,<sup>4</sup> for which Figure 2 presents a schematic diagram.

We can consider this double loop as a "zoomed-in" portion of the system of activity regulation (Figure 1). The object of the action is what is retained in the situation: The comparison between the intended state of this object and the observed effect releases an adjustment of the action. The feedback on the subject (which was not made explicit in Figure 1) will modify the action "upstream" through an adaptation of knowledge and schemes for action. Moreover, inasmuch as there is an intended or anticipated result, the action is also regulated proactively ("feedforward"). Piaget defines this mechanism in terms of a dialectic between *assimilation* of the new situation into the subject's strategies and conceptualizations, and *accommodation* of these concepts and of their organization. (We can think of the passage from a one-dimensional treatment of objects to a bi-dimensional treatment, for which the model is the Cartesian product.)

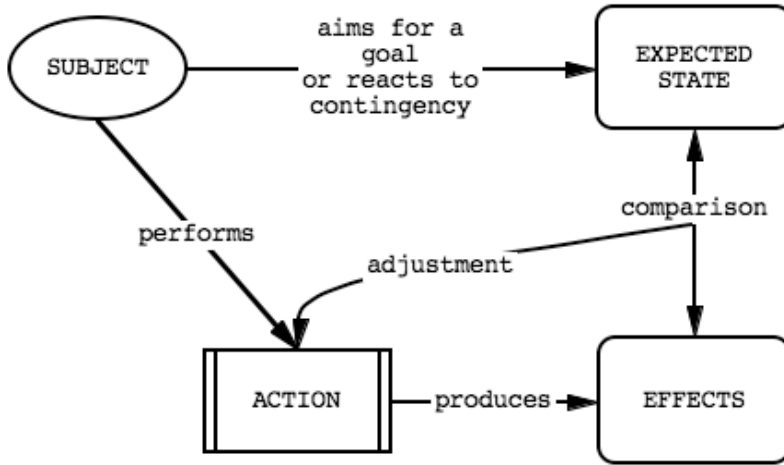


Figure 2. Schematic diagram of action regulation. The regulation takes place within a short action-adjustment loop, dependent on the comparison of the produced effect to the objective of the action. The objective can be determined through a conscious goal or can be a byproduct of the subject getting the object in his field of attention.

The development of knowledge structures results, therefore, from a double process: a *dis-equilibration* when prior knowledge structures lead the subject to expect a result that is invalidated by the action, and a *re-equilibration* after the knowledge structures are modified (Piaget, 1985). Piaget does not say that the re-equilibration is necessarily an improvement, which is to say it does not necessarily lead to a more efficient conceptualization (where “conceptualization” refers to the construction of concepts to understand and act on the world). He also describes the importance of reflective abstraction upon the subject’s activity itself, and not simply on its results. This concept of reflective abstraction was operationalized for mathematics teaching by Simon et al. (Simon, Tzur, Heinz, & Kinzel, 2004).

Within the framework of Piagetian genetic epistemology, the importance of considering knowledge content was reaffirmed by Greco (a collaborator of Piaget who should be rediscovered) and included by Vergnaud in his theory of conceptual fields (see below). This consideration also leads to an analysis of the double regulation as simultaneously a “functional” adjustment of representations of the situation and of the organization of actions, and a “structural” regulation that modifies the conceptual organization and the cognitive operations of the subject. The diagrams in Figures 1 and 2 do not yet differentiate the two types of regulation, or the timeframes in play. We will return to these topics later on.

*Piagetian genetic epistemology: The role of knowledge content*

The Piagetian concept that has probably entered the most into psychology and science of education is that of the stages of development (sensorimotor, concrete operational, formal operational, and eventually intermediary stages as well). The other well-known concept is the idea of a “child logic,” that evolves toward a scientific logic that will permit coordinated logical operations and manipulation of abstract propositions, independent of their content. In fact, the initial work of genetic epistemology on space, numbers, speed, time, and physical concepts presents successive organizations of a child’s representations in each of these domains: these organizations run with a similar underlying structure.

Greco highlighted the role of content in a set of analyses brought together in a posthumous work (Greco, 1991). He provided evidence as to the importance of considering the object of the action, as well as the task. He recalls, “Within the genesis of elementary logical structures, Inhelder and Piaget ... indicate steps or regular levels, but also [insist] on the fact that these steps are strongly differentiated by the nature of the material, the classifying task proposed to the child ... etc.” (op. cit., p. 38). He underlines the necessity of “specifying the conditions of equilibration, notably as these conditions also highlight properties of objects and tasks” (p. 39), and insists on the role played by the object of action and knowledge within the regulation that is at the heart of the development process: “The adjustment of forms to content requires a revision that reveals properties the available forms do not allow us to cover...The worrying question of restoring the role of the object in development is an integral part of Piagetian constructivism” (op. cit., p. 55).

*The complementary contribution of Vergnaud’s conceptual fields*

As highlighted by Greco, the essential Piagetian concepts of dis-equilibration/re-equilibration, the assimilation/accommodation dialectic, and the intervention of a complex regulation process are general concepts that should be made more specific according to the knowledge content. It was Vergnaud who enlarged the Piagetian framework in theorizing the concept of the conceptual field (Vergnaud, 1982, 1990), by outlining situation classes, operational invariants, schemes for action, and representational systems, all relative to a knowledge domain.

The theory of conceptual fields “was initially elaborated in order to take into account the process of progressive conceptualization of additive structures, multiplicative structures, [and] the relationships between number and space, within algebra” (Vergnaud, 1990, p. 135). The theory articulates two epistemological approaches: That of mathematical didactics and that of developmental cognitive psychology. Broadly, from a didactic point of view, the concept of the conceptual field aims to provide a framework for analyzing the student-knowledge relationship within the didactic triangle of student, knowledge, and teacher. From the point of view of developmental psychology, the theorization in terms of



conceptual field permits a joint analysis of the effects of learned concepts and of development throughout a student's mathematical education.

### *The notion of conceptual field*

A typical example of conceptual field is that of additive structures within elementary calculus. Included with this field is a set of numerical concepts: numbers (natural numbers, initially "small" numbers), order relations (first between whole numbers, and then generalized to the number line), and addition and subtraction operations (including their properties, as well as their relationships with order relations and with classes of problems that use these operations for their solution). Trying to consider the concept of number "in itself," isolated from other concepts that render it operational, is not only ineffective for understanding what students learn, but devoid of meaning for studying teaching processes aimed at numeric concepts. The concept of conceptual field is relevant to studying student learning at a wide variety of levels, from the "everyday" conceptualization of a young child (outside of a didactic project), to the conceptualization that is the goal of a scientific lesson or of a student specializing in mathematics.<sup>5</sup>

Thus, at an elementary level, the construction of a complete collection of unique colored shapes, given a set of shapes and an independent set of colors, brings into play a set of concepts related to the conceptual field of the Cartesian product (Rogalski, 1985). These concepts include those of identity and difference for ordered pairs, and eventually the cardinality of sets and the distributive law (ensuring that each form is associated with each color, or vice versa). We note that the concepts in question, while "everyday concepts," are precursors of logico-mathematical concepts. Another example of such everyday concepts is the "small" cardinals, which a small child is able to manipulate after mastering its precursor, "numerosity," a quality perceived in object collections, not unique to humans but shared by a number of species. All these concepts will continue to develop and enter into the conceptual fields of additive then multiplicative structures.

In general, concepts can be considered as nodes of a network that contains all kinds of relations. A conceptual field corresponds to a part of this network that possesses characteristics relevant to a set of proposed situations. Thus, a number of concepts are contained within the conceptual field of numeric multiplicative structures, including the Cartesian product, product measure, linear transformations on  $\mathbf{R}$ , multiplicative operations on various numeric sets ( $\mathbf{N}$ ,  $\mathbf{D}$ ,  $\mathbf{Q}$ ,  $\mathbf{R}$ ). Similarly, a single scientific concept can fall under several different conceptual fields. The concept of surface area, for example, is included in the conceptual fields of physical quantities, measure operations, sets of positive numbers, space and its models, it belongs to the conceptual field of additive structures (as a "simple" measure), and of multiplicative structures (as a product of linear measures).

### *Schemes in the theory of conceptual fields*

Vergnaud insists on the fact that, to take into account the adaptive function of knowledge, it is necessary to give a central place to the operational dimension of knowledge ("rational knowledge is operational or nothing," op. cit., p. 136). The



concept of a “scheme” models this operational function. With Piaget, the scheme is defined as an invariant organization of action for a class of situations.

It should be stressed that it is not the action that is invariant, but a particular property of the action: its organization. The operational nature of the scheme reflects the possibility that the action may vary with the determinants of the subject’s situation. This is what enables new situations to be met with adaptation rather than simple repetition. In the Piagetian line of research, a number of processes have been proposed for the development of schemes: a double process of assimilation and accommodation (similar to the process in play in conceptual development), processes of generalization/specification, and processes of combining pre-existing schemes.

Within the domain of learning mathematics, Vergnaud described the evolution of a counting scheme during the numeric education of children and students. This scheme involves the temporal coordination of visual focus, pointing gestures, and recitation of the list of number names, and repetition of the last word-number (an ordinal) to give the cardinality of the set (“one, two, three ... seven – SEVEN!”). The child will assimilate new counting situations into a scheme initially developed for very small collections. New situations will call for an accommodation in the initial scheme. For “big” collections, operations will be added to the scheme’s organization. These may include taking into account the spatial structure of the collection, using the theorem-in-action of adding the cardinals of disjoint subsets, or using an intermediate system (tally marks grouped in fives, used in manual vote counting systems, for example). The “technical” operations of addition will stem from this.

The analysis proposed in the theory of activity, together with “activity, action, operation” (Galperine, 1966; Haenen, 2000; Leontiev, 1984; Savoyant, 1979, 2005) leads to a distinction of levels within schemes.<sup>6</sup>

The issue of schemes of action for the student is considered in studying the development of what we could call their dexterity in executing mathematical procedures, which falls outside the scope of this book.

### *The importance of systems of representation*

Within the theory of conceptual fields, developing “a psychological and didactic approach to the formation of mathematical concepts leads to a consideration of a concept as a set of invariants that are available for use in an action. The pragmatic definition of a concept, therefore, includes the set of situations which constitutes the reference for the various properties of the concept, as well as the set of schemes applied by subjects in these situations. However, the operative action does not constitute the entire conceptualization of reality ... the use of explicit signifiers is indispensable to the conceptualization” (op. cit. p. 145). Inhelder and Piaget’s declaration that “memory of a scheme is that scheme” brings us back to the issue of progression from schemes’ “concepts-in-action” and “theorems-in-action” to representable concepts, following a conscious realization.

For Vergnaud, representations are two-sided: The “signifier” corresponds to their external dimension, and the “signified” to their internal dimension. External

representations (signifiers) can take a variety of forms. Vergnaud, like Vygotsky, highlighted the central place of verbal language (as opposed to non-verbal; can be oral or written). The treatment of symbolic representations can be an intrinsic part of the activity. This is a crucial point for analyzing the double teaching/learning process within secondary education, particularly where algebra is concerned.

*Collective history of mathematics, individual history of the student*

At the moment of teaching, a process of conceptualization has already taken place within “mathematical communities.” This process results in the production of “theoretical knowledge” that is at the heart of epistemological analysis in mathematical didactics. These historically constituted conceptual organizations of scientific knowledge serve as a reference to determining the relevant conceptual fields for analyzing and provoking students’ conceptualizations, issued from their activity in appropriate situations.

Students’ personal history and the familiar frameworks in which they act may introduce new elements in their activity, in addition to the conceptual structure of the situation. Vergnaud elaborated a typology of addition problems in which psychological and didactic relevance is denoted by important differences in student learning. His typology departs of the mathematical models that essentially limits variation to the numeric values and to the “technical” operations available for use in solving.

The “socio-constructivist” theoretical framework elaborated by Vygotsky enables taking this historical double determination into account, and outlines two concepts. The first concept is that of *social mediation*. This concept primarily intervenes through a direct intervention by an adult or a “more knowledgeable” into the activity of the student or child. This analysis was later developed by Bruner, with the goal of developing a theory of instruction (Bruner, 1996; Wood, Bruner, & Ross, 1976). The second concept is the “everyday concepts/scientific concepts” dynamic, where the latter are instruction objects, analyzable as “theoretical knowledge.”

This Vygotskian framework also allows for consideration of the “learning/development” process as one point of view on the subject, and for expanding the issue of development (which psychology and Piaget himself, traditionally limited to childhood) to cover the entire lifespan of the subject.

After recalling, below, Vygotsky’s contributions, we will put into perspective the Piagetian and Vygotskian frameworks as compatible and complementary tools for analyzing teaching practices and student learning.

*Vygotsky’s views of didactic intervention and of development*

Vygotsky’s theory put social mediation, and the value of “psychological tools” in this mediation, at the heart of the development process. It thus offered a perspective that was complementary to Piaget’s for studying the teaching/ learning relationship within the development of the conceptual fields of a scientific domain such as mathematics. Vygotsky also contributed to the analysis of

conceptualization, showing how “everyday” concepts from the child’s normal life, and “scientific” concepts that were explicitly taught, developed within a “double germination” dialectic.

*Everyday concepts and scientific concepts*

Chapter 6 of *Thought and Language* (Vygotsky, 1986) explores the topic of two types of concepts: everyday concepts and scientific concepts. It also describes the relationships with the learning and development of a child. Vygotsky criticizes Piaget for only being interested in the development of spontaneous or everyday concepts, and for not examining the form scientific (“taught”) concepts took in a child or adolescent, and how they were integrated into development.

The distinction introduced here by Vygotsky contrasts the characteristics of concepts stemming from a child’s interactions with objects in the world without didactic intervention, with characteristics of scientific concepts originating from a prior collective production, which are also objects of instruction.

Two characteristics distinguish everyday and scientific concepts: Their organization (the “structural” dimension) and their relationships with objects (the “functional” dimension). Scientific concepts are strongly tied each other by mutual relationships, including abstraction (generalization) relations, while everyday concepts can be isolated. In the activity of a very young child, the concept of *cat* is not necessarily tied by an abstraction relation to the concept of *feline* or *mammal*. Nor must it be placed in comparison to, or contrast with, the concept of *dog*. The *cat* concept is functional, and operational, without any such relationships. By contrast, the concept of *function* in mathematics (numerical, one-variable) cannot exist without that of *variables*, while the operationalization of the concept of variables assumes that of *numeric concepts*. In addition, the concept of *graph of a function* (a one-dimensional subset of the plane) and its graphical (external) representation implies a *number-space relation*, with the concepts of *number line*, *x-axis*, *y-axis*, and *coordinates*.

Everyday concepts can exist “in action” without children being conscious of them or able to verbalize them, either because the process of consciously realizing the “concept-in-action” did not take place during the child’s development, or because the action does not call it to mind. By contrast, scientific concepts are explicit and exist through symbolic representations, such as language (the primary form of representation) and other symbolic mathematical systems.

Vygotsky presents the idea that an everyday concept is “glutted with empirical content.” This is a strength of everyday concepts from the point of view of significance, but it is also a weakness, as the content brings with it a mass of properties, which limit conceptual constructions at a higher abstract level.

By contrast, the strength of a scientific concept comes from its generality in terms of abstraction, and the generality of the domain in which it acts (its “decontextualization”). Its strength also comes from the fact that the concept is conscious and was constructed with “words to say it,” as well as from the coherence of the system of concepts to which it belongs. However, despite being

more general than everyday concepts, scientific concepts display much less empirical concept, which is their weakness, in terms of significance.<sup>7</sup>

*The dialectic of “double germination” of everyday concepts and scientific concepts*  
The respective characteristics of everyday and scientific concepts can lead to considering them as two conceptual categories. The historical-developmental dimension of Vygotsky’s theoretical approach is in fact essential for progressing past this view, to seeing everyday and scientific concepts as developing interaction within a dialectic of “double germination.”

In one direction, the germination of scientific concepts passes from the “low” to the “high,” where the “high” is what is “general” and “decontextualized.” This passage follows interaction with objects from the world of action (as in Piagetian constructivism). In another direction, this germination passes from high to low, supported by symbolic representations (including appropriate language) proposed in the mediation.

Within this biological metaphor of germination, everyday concepts clear the way for the germination of scientific concepts by the meaning they provide. Scientific concepts, in turn, clear the way for the germination of everyday concepts through their organization and the mediations they propose, and “pull” the everyday concepts higher.

In terms of the child, an operational piece of knowledge is acquired when the two types of concepts meet and two processes are engaged. The first process is a reorganization of everyday concepts to better organize them into a system. The second process involves extracting the meaning of scientific concepts to make them concepts for action. This process of interaction assumes a property of the development dynamic: that the dynamic takes place within a *proximal development zone*.<sup>8</sup>

*The proximal development zone and the learning/development relationship*

The proximal development zone (PDZ) is situated between the current level of development, defined by what the child is capable of doing or solving autonomously, and what the child can do or solve with the help of others (adult, teacher, more knowledgeable peer). For conceptual learning to succeed, situations should raise this zone. If they are above the PDZ, assistance can at best produce an immediate imitation (or recitation; Vygotsky speaks of “verbal mechanics”), and do not contribute to development. If the situations fall below this zone, the child/student only uses prior knowledge, and learns nothing.

The concept of PDZ is relevant for the initial development of a new conceptualization that is based on prior mathematical knowledge acquired by the student. By acting on the student in this zone, the teacher allows everyday concepts, or familiar mathematical concepts, to transform and integrate into a more detailed conceptual field. If the mediation is successful (assuming its goal is conceptualization and not the simple mastery of procedures), these concepts can go toward the mathematical concepts to which they are epistemologically tied.

The Vygotskian concept of PDZ is not only a way to discuss development/learning relationships. It transforms the understanding of these relationships. Under the Vygotskian theory, scientific conceptualization does not await, rely on, or follow spontaneous conceptual development, but instead intervenes in this development, offering new mediations through new tools for thinking. As for the learning dynamic, it too depends on the possible meeting of taught concepts and meaning brought through everyday concepts (or familiar mathematical concepts), even if the latter are weaker in terms of their generality and organization. Vygotsky thus proposes a dialectic process of conceptualization at the heart of the development processes.

### *Merging the Piagetian and Vygotskian frameworks*

We have described the respective approaches of Piaget and Vygotsky from an epistemological point of view, which has led us to consider them non-contradictory. In addition, we have examined the parts of their respective theoretical frameworks concerning conceptual development, in terms of issues of mathematical didactics (and other didactics centered on conceptualization). These considerations lead us to highlight their commonalities, or complementarity, where Bruner “celebrated their differences,” arguing that “with one thinker emphasizing the role of inner autochthonous logical<sup>9</sup> processes, and the other the shaping role of culture, inevitably led to sharp divergences in their approach of mental growth” (Bruner, 1996). Here we take the same path as other researchers, such as Cole and Wertsch, specialists in Vygotsky who proposed going beyond the apparent social/individual incompatibility between the two (Cole & Wertsch, 2001). In fact, Piaget never denied the key role of the social dimension in child development, but simply did not include it in his theory. Looking at Vygotsky, his socio-constructivist theory is in no way incompatible with the concept of structuration of subject knowledge via the regulation processes of the Piagetian framework, even if he did not specifically consider these processes. The following discusses their commonalities in terms of factors of development, psychological or cognitive tools, and long-term development.

### *Factors of development*

We highlighted above the fact that even if factors of development were not at the heart of Piaget’s genetic epistemology, he nevertheless did not reduce them to interaction with the objects of the action. He stressed three general factors: biological maturation, the role of exercise and experience gained in the action, and, finally, social interactions and transmissions (Piaget & Inhelder, 1971, p. 152 ff.). The social dimension of Piaget’s theory was studied more generally by DeVries (1997). As for Vygotsky, he made explicit the role of interaction with objects of the world of action within a child’s development. Vygotsky’s socio-constructivism is thus a materialist constructivism.

*Psychological tools, cognitive tools*

One central element in Vygotsky's theory is the role of tools in a child's development. He focuses particularly on psychological tools, as these have already been constructed socially. A very specific place is given to language, a psychological tool *par excellence*.

Piaget largely used language (and graphical representations) as a way to access a child's "spontaneous representations." Even so, he did not question the role of language in development, but explicitly referenced the contributions of language to cognitive tools: "... language has already been elaborated socially and contains a notation for an entire system of cognitive instruments (relationships, classifications, etc.) for use in the service of thought. The individual learns this system and then proceeds to enrich it" (Piaget & Inhelder, 1971, p. 87).

*The long-term development of a subject*

The study conducted by Piaget and his collaborators regarding stages of cognitive development for the large categories of thought regularly stressed the long-term nature of this development. Within the numerical domain, Vergnaud showed that the conceptual field of additive structures develops precociously, with additive operations appearing at two years, while transpositions and comparisons<sup>10</sup> were not mastered until the end of mandatory schooling. Data on learning spatial measurements, particularly volume, have shown the difficulty and length of the conceptualization process required to progress from a "one-dimensional" understanding of volume (a familiar concept at the end of elementary school) to the idea of a product measure (for which conceptualization is not achieved at the end of junior high school).

Vygotsky did not have the same insistence on the long-term view of cognitive development, but he did stress, in the chapter entitled "Everyday concepts, scientific concepts" (1986), that the construction of scientific concepts, like everyday concepts, only began after the child had assimilated for the first time a new meaning or term, bringing with it a scientific concept.

*Didactics and extending the timeframe of development*

The didactic interpretation of "the two constructivisms" has led us to extend the timeframe of development to "advanced math." The concepts already available within a conceptual field are potential *precursors* of the concepts to be learned, and have a function analogous to that of everyday concepts in the Vygotskian framework. These precursor concepts have two possible and contradictory roles, that of a precursor and that of an obstacle. The productive role of precursors is, in particular, to give meaning to new concepts (Vygotsky spoke of the "force" of everyday concepts). Their reductive role is tied to properties of concepts that are no longer valid for the new concepts to be learned (following Bachelard, French didactics calls these "epistemological obstacles"). In Piaget's theory, the duality of the productive/reductive roles can be interpreted in terms of the interplay between the processes of assimilation and accommodation.

One important contribution of the Piagetian framework, for which we find no Vygotskian analogue, is the fact that knowledge develops through a “Münchhausen effect”: “One of the strength-ideas of Piagetian constructivism is that knowledge itself creates the conditions and tools of knowledge” (Greco, 1991, p. 52). There is, thus, a dynamic unique to knowledge beyond occasions of development provoked by didactic mediation.<sup>11</sup>

*The two constructivisms: complementary theoretical tools*

Each of the two main figures in constructivism recognized the strengths of the other. Vygotsky, discussing the shift toward scientific concepts of a certain number of notions of causality, remarked, “I have not looked closely at the state of children in terms of the logic they are capable of using; in this Piaget has shown overwhelming superiority” (Vygotsky, 1997). As for Piaget, he noted his agreement with Vygotsky’s approach to the analysis of everyday and scientific concepts. In addition, he stressed that “The individual only achieves intellectual constructions insofar as he is the seat of collective interactions, whose level [depends] on the society of his group” (Piaget, 1992, p. 345).

Finally, a more philosophical point of agreement between the Piagetian and Vygotskian frameworks was highlighted by Bruner (1996): “The unique mystery of mind is its privacy, its inherent subjectivity. Both Piaget and Vygotsky were very explicit on this point. See Piaget (1974, pp. 28 ff.); Bruner’s (1987) preface to Volume One of Vygotsky’s collected works.” This concordance on the subjectivity of thought reinforces the links between the two developmental frameworks and the theory of double regulation of the activity. The latter concentrates on subjects, authors of and actors in their own activity, whether they are students whose learning is the goal, or teachers who work toward student learning.

CONCLUSION: OUTLINING THE THEORY OF ACTIVITY  
AND THE TWO CONSTRUCTIVISMS

We presented the theory of activity and the model of double regulation of the activity, which is used in ergonomic psychology but is extendable to any completed activity, including that of the student. One component of the double regulation model is the impact of the activity on subjects themselves, which represents the developmental dimension of this model. On the topic of knowledge, we then highlighted the commonalities and complementarity of the constructivist theories of Piaget (as extended by Vergnaud’s conceptual fields) and Vygotsky (with Bruner’s theory of scaffolding).

The connection between the theory of activity and the “two constructivisms” thus offers a theoretical tool for a double approach from the viewpoints of mathematical didactics and the activity of the subjects in question (teacher and students). In particular, the Piagetian theory looks “from the student’s side” at epistemological analyses of the mathematical objects in play, while the Vygotskian theory takes into account the didactic intervention of the teacher, mediating between knowledge and student in support of the student’s activity.



The developmental dimension also calls into question the timeframe of the processes in play, particularly for students. Leplat already compared the theory of activity and Piagetian constructivism and highlighted the existence of a functional regulation as well as a structural regulation. The functional regulation leads to adjustment of the action (cf. Figure 2), within a process of micro-genesis, that can translate through procedural learning, with possible support from the teacher. This short-term regulation can also involve conceptual “primings” within the student’s PDZ, between the previously acquired concepts and new mathematical concepts. The structural regulation acts over the long-term, within a process of macro-genesis, in which the conceptual structures and the student’s schemes of action are transformed within an assimilation/accommodation dialectic.

From a didactic perspective, we can take the relationship between these two timeframes into account by forming the hypothesis that after concepts are “primed” (whether through a fundamental situation, an appropriate problem, or even an explicit direction from the teacher), making these concepts functional is an essential contribution to the structure of the intended conceptual field<sup>12</sup>. The methodological importance comes partly from analyzing the mathematical tasks proposed by the teacher in light of the potential student activity during the completion of these tasks, and partly from analyzing the didactic interventions on the activity of students in class. Two delicate elements within analyses of teaching/learning situations are the role played by the autonomous activity of the student, and the importance of the teacher’s identification of the PDZ.

There are some very general avenues for research that rely on the theoretical tools presented here. There are still aspects to be clarified enabling the study of didactic interventions and student development at various levels of analysis: from the global level, consisting of relationships between the overall structures of teacher interventions and the mathematical knowledge acquired by students<sup>13</sup>, to the “micro” level of individual interactions during class, passing through the local level of a class period.

## NOTES

<sup>1</sup> By “competencies,” we refer to the sense as it is used in ergonomic psychology and professional didactics. This type of competency does not denote a set of tasks that can be completed successfully, but a set of potential resources for action of a subject. It is the same sense intended when referring to a student’s competency in mathematics.

<sup>2</sup> Note: This task-activity distinction differs from formulations encountered in various pedagogical texts. Within the theory of activity “The activities proposed to a student” would be expressed as “The tasks proposed ....”

<sup>3</sup> Berliner (2001) showed the complexity of the question of the characteristics and even definition of an “expert” teacher. A description of approaches that agrees with the one we propose here is presented by Perrenoud (2005), with regards to the question of knowledge mobilized in the analysis of teaching practices.

<sup>4</sup> This conception should be linked to its importance for cybernetic concepts, for which regulation is a central concept, from biology to automated systems.

<sup>5</sup> The entry point chosen for analysis here is that of concepts. This entry point is directly applicable for studying student learning in scientific disciplines (the conceptual fields are here defined with



reference to theoretical knowledge). A “dual” entry point by way of situations is used in professional didactics, with the notion of “conceptual structure of a situation,” describing diverse concepts, including pragmatic ones (Pastré, 1999; Vidal-Gomel & Rogalski, 2007). Brousseau’s concept of the fundamental situation could be seen as an expression of the epistemological link between concepts and situations.

- <sup>6</sup> The analysis is complicated by the fact that these levels are relative. An action previously composed of multiple operations can become, during development, a “unitary” operation that is itself a component of a higher-level action.
- <sup>7</sup> Within the work domain, professional didactics introduced the theoretical notion of “pragmatic concepts” as the organizers of the activity. Historically constructed by a professional community within and for a particular domain, these are neither “everyday” concepts nor “scientific” concepts (or techniques) under Vygotsky’s definition. Integrated within a conceptual structure of the work situation, these concepts relate to indicators (observables) and ways of acting (Vidal-Gomel & Rogalski, 2007). In teaching, an expression such as “the class has disengaged” refers to a pragmatic concept for which teachers use various indicators and have a multitude of possible interventions (changing tasks, intervening in students’ activities, etc.).
- <sup>8</sup> A number of versions of this concept can be found in the literature: zone of near development, zone of proximal development, or even zone of potential development. I have chosen to use “proximal” to refer to this zone.
- <sup>9</sup> Our focus, coming from didactics, is actually on the development of knowledge and particularly the process of conceptualization, rather than on general logical processes.
- <sup>10</sup> Jean-François Richard (2004) showed that humanities students encountered serious problems in descriptive statistics related to issues of cumulative effects (requiring them to perform subtractions). My own experience has taught me that we find these types of problems in errors on credit and debit in accounting.
- <sup>11</sup> It is the existence of the dynamic unique to knowledge that led us to consider the activity of the teacher as the management of a dynamic environment, which is the student/knowledge relationship (Rogalski, 2003).
- <sup>12</sup> We can even make the hypothesis that making concepts functional can create meaning, under certain conditions on the density of work and the position of the student in relation to mathematics.
- <sup>13</sup> The methodological problems of defining adequate global indicators, as much for student learning (beyond assessments of success at certain types of tasks) as for relevant properties of the teacher’s intervention, remain open.

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